Skip Lester
Math 152

Chapter 5

5.1 Areas and Distances

The Area problem: How do we determine the area between a function and the x-axis over a given interval?

We divide the interval into rectangles of varying height and approximate the area by summing the areas of the rectangles

Estimating the area under x^2

Placing the left endpoints of our rectangles on the function yields an underestimate of the area under x^2 over the given interval. Placing the right endpoints of our rectangles on the function yields an overestimate. The true area must be between the two estimates.

Now, increase the amount of rectangles, creating more ever-thinner strips, and repeat the process of summing their areas. As the number of strips increases, so too does the precision of the answer. The answer approaches $\frac{1}{3}$ as the number of rectangles approaches infinity.

The areas of the rectangles are $\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2+\left(\frac{1}{n}\right)\left(\frac{3}{n}\right)^2+\ldots+\left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$

Factoring out $\left(\frac{1}{n}\right)^3$, we have $\left(\frac{1}{n}\right)^3\left(1^2+2^2+3^2+\ldots+n^2\right)$

To solve, we replace $\left(1^2+2^2+3^2+\ldots+n^2\right)$ with the sum of squares for the first positive n integers, where n represents the number of strips we have divided our interval into.

Now we have $\left(\frac{1}{n}\right)^3\left(\frac{n(n+1)(2n+1)}{6}\right)=\frac{(n+1)(2n+1)}{6n^2}$

All that remains in order to determine the area is taking the limit as *n* becomes large positive.

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}=\frac{1}{3}$$

Notation

Since the area under the graph of a continuous function is a limit and a sum, it is important to understand **sigma notation**. The area under the continuous function *f* can be described by

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(\triangle x)$$

Distance

To find the distance travelled by an object during a given interval of time, we must know the velocity of the object. If the object has a varying velocity, it can be described with a function. Taking the integral of that function, which represents velocity, will give us the distance the object has travelled. Polling for sample points more frequently yields higher accuracy.

$$\text{distance} = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)(\triangle t)$$

5.2 The Definite Integral

Definition $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

The function *f* being integrated is called the integrand. The integrand can take on negative values; if *f* has negative area, then the definite integral is the sum of the positive area and negative area. The net area is therefore the area below the x-axis subtracted from the area above it on the interval [a,b]. The function must be continuous or suffer only a finite number of jump discontinuities in order to be integrable.

Evaluating Integrals

Sum of Positive Integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Sum of Positive Integer Squares: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of Positive Integer Cubes: $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Sigma Notation rules:

$$nc = \sum_{i=1}^{n} c \qquad \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$c\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ca_i \qquad \sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

The Midpoint Rule

When approximating an integral, we use midpoint for sampling to get the most accurate estimate.

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + \ldots + f(\overline{x}_n)]$$

Where $\Delta x = \dfrac{b-a}{n}$ *and* $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i) =$ *midpoint of* $[x_{i-1}, x_i]$

Properties of the Definite Integral

The definition of the integral was introduced such that a < b. However, the definition is resilient even if a > b. If a>b, then $\Delta x = {(a-b)}/{n}$ instead of ${(b-a)}/{n}$.

Intuitively, if a = b, then $\Delta x = 0$, and the area = 0. $\int_a^a f(x)dx = 0$

True for a <b, a = b, and a>b:

1. $\int_a^b c\,dx = c(b-a),$ *where c is any constant*
2. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx,$ *where c is any constant*
4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ *(adjacent intervals)*

True only for $a \le b$:

6. *If* $f(x) \ge 0$ *for* $a \le x \le b$, *then* $\int_a^b f(x)dx \ge 0$.

7. *If* $f(x) \ge g(x)$ *for* $a \le x \le b$, *then* $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

8. *If* $m \le f(x) \le M$ *for* $a \le x \le b$, *then* $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

5.3 Evaluating Definite Integrals

Evaluation Theorem

If *f* is continuous on the interval [a,b] , then

$\int_a^b F'(x)dx = F(b) - F(a)$, where *F* is any antiderivative of *f*. That is, *F’* = *f*.

$$\int_a^b f(x)dx = F(x)]_a^b$$

Table of Indefinite Integrals

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx \qquad \int cf(x)dx = c\int f(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \ne -1) \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \csc x \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

Applications

Net Change Theorem

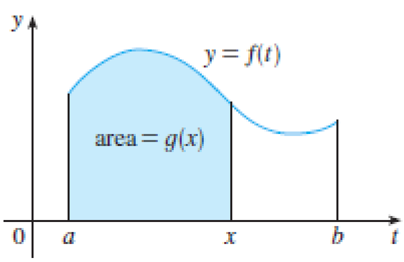
The integral of a *rate of change* is the **net change**:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

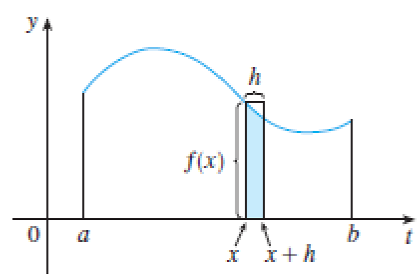
5.4 The Fundamental Theorem of Calculus

The "Area So Far" Function

If *f* is continuous on the interval [a,b] , then the function *g* defined by



$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$
$$= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt$$
$$= \int_x^{x+h} f(t)dt$$
$$\therefore \text{ for } h \neq 0,\ \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$



is an **antiderivative** of *f*, that is, *g'(x) = f(x)* for a<x<b.

Alternatively, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Combining the Chain Rule: U subsitution

Find $\frac{d}{dx}\int_1^{x^4} \sec t\, dt$: Let $u = x^4$. Then $\frac{d}{dx}\int_1^{x^4} \sec t\, dt = \frac{d}{dx}\int_1^{u} \sec t\, dt$

$= \frac{d}{du}\left[\int_1^u \sec t\, dt\right]\frac{du}{dx}$ (by the chain rule) $= \sec u \frac{du}{dx} = \sec(x^4)(4x^3)$

Differentiation and Integration as Inverse Processes

Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)dx = F(b) - F(a)$,

where F is any antiderivative of f, that is, $F' = f$.

Proof of the Fundamental Theorem of Calculus

$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

$$= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt$$

$$= \int_x^{x+h} f(t)dt$$

$$\therefore \text{ for } h \neq 0,\ \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

5.5 The Substitution Rule

Introducing Something Extra

We change from the variable *x* to a new variable *u.* We let *u* represent a function of *x* and replace *dx* with *du*

U Subsitution

Evalulate $\int 2x\sqrt{1+x^2}\,dx$ let $u = 1 + x^2; du = 2x\,dx$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{(3/2)} + C = \frac{2}{3}\left(x^2+1\right)^{(3/2)} + C$$

U Substitution related to the Chain Rule

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \text{ because } \frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

$\therefore$ If $u = g(x)$ is a differenetiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

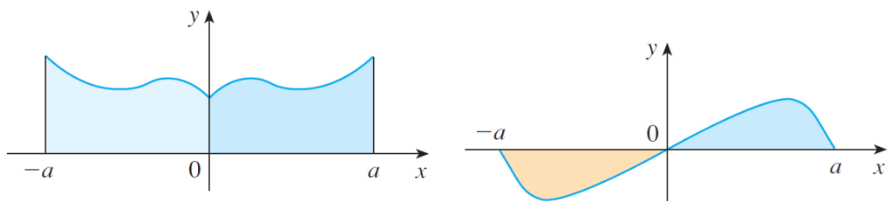
$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Symmetry

Suppose f is continous on $[-a,a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0.$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

5.6 Integration by Parts

Integration by Parts is the Corresponding Integration to the Product Rule

Recall that the Product rule of Differentiation gives

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Thus
$$\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x)$$
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Formula for Integration by Parts

Let $u = f(x)$ and $v = g(x)$

The differentials are

$du = f'(x)dx$ and $dv = g'(x)dx$

By the Substitution Rule,

$$\int u\,dv = uv - \int v\,du$$

Sometimes you may need to Integrate by Parts twice!

With the Evaluation Theorem

$$\int_a^b f(x)g'(x)dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

5.7 Additional Techniques of Integrations

Trigonometric Integrals

Evaluate $\int \cos^3 x$.

To integrate powers of cosine, we need an extra $\sin x$ factor.

By $\sin^2 x + \cos^2 x = 1$, we rewrite the integrand

$\cos^3 x = \cos^2 x(\cos x) = (1 - \sin^2 x)(\cos x)$

Next, substitute $u = \sin x$, so $du = \cos x\,dx$.

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x(\cos x)\,dx = \int (1-\sin^2 x)(\cos x)\,dx \\ &= \int (1-u^2)du = u - \tfrac{1}{3}u^3 + C \\ &= \sin x - \tfrac{1}{3}\sin^3 x + C\end{aligned}$$

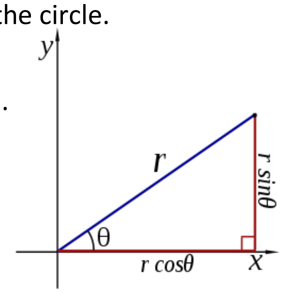
Trigonometric Substitution

Prove that the area of a circle with radius r is πr^2.

Setting our circle's center to origin, we have $y = \pm\sqrt{r^2 - x^2}$

By symmetry, the area in quadrant I is equal to one fourth of the total area of the circle.

Now we have $y = \sqrt{r^2 - x^2}$ $0 \le x \le r$, thus $\frac{1}{4}A = \int_0^r \sqrt{r^2 - x^2}\,dx$.



By the unit circle, $x = r\sin\theta$. $\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\sin^2\theta}$

Again using the helpful identity $\cos^2\theta = 1 - \sin^2\theta$, we rewrite

$$\sqrt{r^2(1-\sin^2\theta)} = \sqrt{r^2\cos^2\theta} = r\cos\theta$$

When we substitute a new variable, we must equate the limits of integration

$$\int_0^r \sqrt{r^2 - x^2} = \int_0^{\pi/2} (r\cos\theta) r\cos\theta\, d\theta = r^2\int_0^{\pi/2}\cos^2\theta\, d\theta$$

By $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we have $r^2\int_0^{\pi/2}\cos^2\theta\, d\theta = \frac{r^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta$

$$= \frac{r^2}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{r^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi r^2}{4} \square$$

The identity $\sin^2 x + \cos^2 x = 1$ is useful in rewriting the integrand so that we have only sine factor and the rest of our terms in cosine (or vice versa).

Partial Fractions

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

1. Turn the integrand into a proper fraction
2. Factor the denominator completely
3. Set up the partials by writing each factor's degree in successive powers in separate fractions

e.g. $\dfrac{A}{(x+1)^1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{(x^2+3)}$

4. Multiply the resultant equation by the least common denominator
5. Create a system of equations by equating the coefficients of like degree terms
6. Solve for the coefficients
7. Replace the coefficients in the partial fraction side of the equation and evaluate the integral

Appendix G: Integration of Rational Functions by Partial Fractions

Formatting

For partial fractions to work, the degree of the polynomial in the numerator must be less than the degree of the polynomial in the denominator. If the degree of the polynomial in the numerator is greater than the denominator, we must perform long division until we arrive at

$$f(x) = \frac{P(x)}{Q(x)}, \text{ degree of } P(x) < \text{ degree of } Q(x)$$

Next we simplify the denominator into its distinct linear or quadratic factors.

Forms

For every term in the denominator, we write on the right hand side of an equation

$$\frac{A}{(ax+b)^i} \text{ or } \frac{Ax+B}{(ax^2+bx+c)^i}$$

So that we have

$$\frac{P(x)}{Q(x)} = \frac{A}{(ax+b)^i} + \frac{B}{(ax+b)^{i+1}} + \frac{C}{(ax+b)^{i+2}} + \frac{Dx+E}{(ax^2+bx+c)^i}$$

Evaluation

In order to find values for A, B, C, D, and E, we now mutliply both sides of the equation by the numerator $Q(x)$. Then we form a system of equations by equating the coefficient of like-degree terms. Solving for the coeffiecents, we fill the values back into the simplified fractional integral, and evaluate it.

5.8 Integration Using Tables and Computer Algebra Systems

Tables of Integrals

Tables of indefinite integrals list forms of integrals and their evaluations.

Typically, algebraic manipulation and substitution is used to transform your integral into one of the listed integrals.

Computer Algebra Systems

Mathematica constantly answers in ridiculous forms. Wolfram alpha does the same. Imperative is the ability to spot equivalent forms. Distribution, forms, and order can all vary, especially when dealing in trigonometry.

Can We Integrate All Continuous Functions?

No.

In single variable calculus, we focus on **elementary functions**:

Polynomials, rational functions, power functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, and functions obtained by addition, subtraction, multiplication, division, and composition with another function.

If a function f is an elementary function, then $f\,'$ is also an elementary function.

However, f a continuous elementary does not guarantee $\int f(x)\,dx$ to be an elementary function.

Most elementary functions don't have elementary antiderivatives. Examples

$$\int \frac{1}{\ln x}\,dx \qquad \int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx \qquad \int \sqrt{x^3+1}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

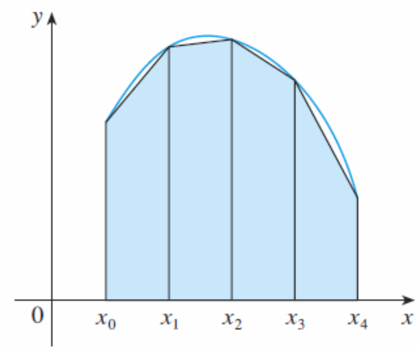
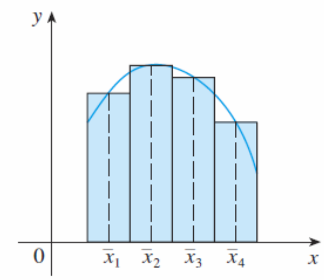


FIGURE 2
Trapezoidal approximation

5.9 Approximate Integration

Reimann Sums

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$



(c) Midpoint approximation

Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + \ldots + f(\overline{x}_n)]$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i) =$ *midpoint of* $[x_{i-1}, x_i]$

Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$

Error Bounds

Suppose

$|f\,''(x)| \le K$ for $a \le x \le b$. Then the errors for Trapezoidal and Midpoint Rules are given by

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

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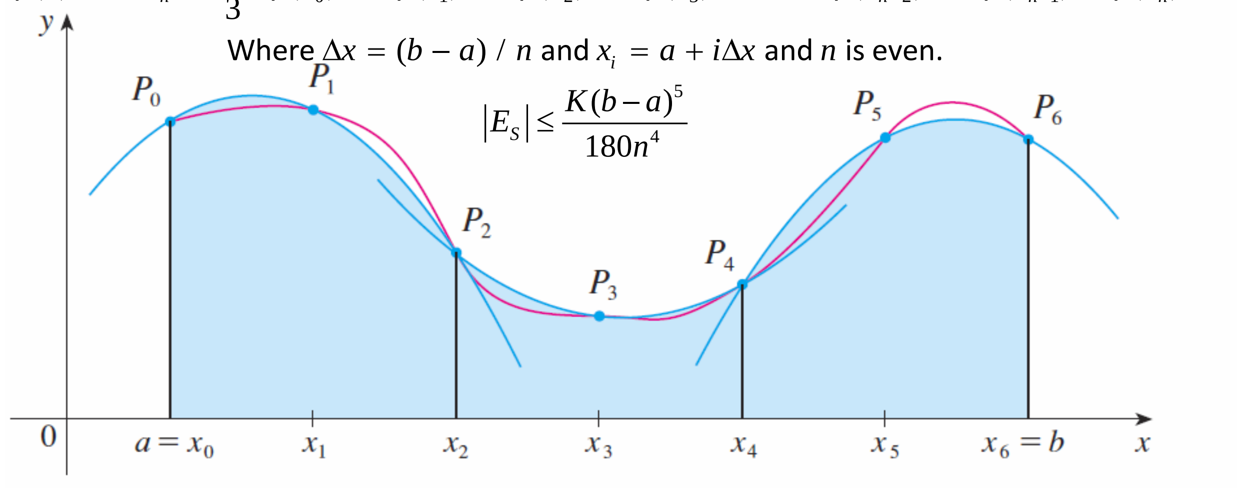
Simpson's Rule

We can approximate integration using parabolas to approximate curves.

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Where $\Delta x = (b - a) / n$ and $x_i = a + i\Delta x$ and n is even.

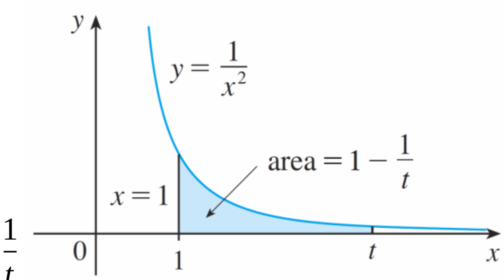
$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$



5.10 Improper Integrals

An improper integral is defined as a definite integral with an infinity limit or an infinite discontinuity on the interval of the definite integral.

Type 1: Infinite Intervals



Intuitively, it seems like the area described by this integral is infinite, because we have infinity in the limits of integration. To go further, we'll need to make a substitution: Let $t = \infty$. The area under the integral is now a function of t.

Consider $\int_1^\infty \frac{1}{x^2} dx$.

$$A(t) = \int_1^t \frac{1}{x^2} dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ for $0 < t$

and $\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$

Thus, $\int_1^\infty \frac{1}{x^2} dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2} dx = 1$

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$, provided this limit exists as a finite value.

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$, provided this limit exists as a finite value.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are said to be **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are **convergent**, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx,\ a \in \mathbb{R}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ is **convergent** if $p > 1$ and **divergent** if $p \leq 1$.

Type 2: Discontinuous Integrands

Type 2 improper integrals deal with discontinuities on the integrated interval, like vertical asymptotes.

(a) if f is continuous on $[a,b)$ and discontinous at b,

then $\int_a^b f(x)\,dx = \lim_{t\to b-} \int_a^t f(x)\,dx$, if this limit exists as a finite number.

(b) if f is continuous on $(a,b]$ and discontinous at a,

then $\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx$, if this limit exists as a finite number.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

A Comparison Test for Improper Integrals

Parts (b) and (c) of Definition 3 are illustrated in Figures 8 and 9 for the case where $f(x) \geq 0$ and f has vertical asymptotes at a and c, respectively.

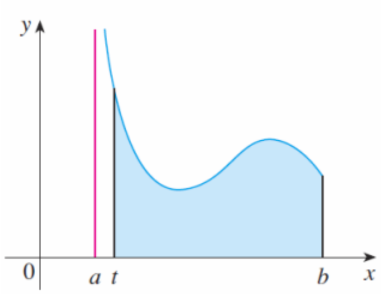


FIGURE 8

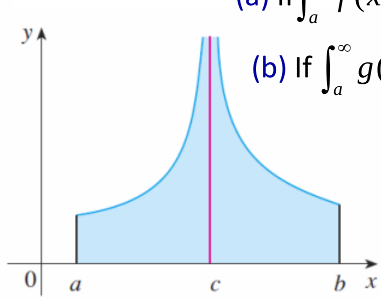


FIGURE 9

Suppose that f and g are continuous functions $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is **convergent**, then $\int_a^\infty g(x)\,dx$ is **convergent**.

(b) If $\int_a^\infty g(x)\,dx$ is **divergent**, then $\int_a^\infty f(x)\,dx$ is **divergent**.

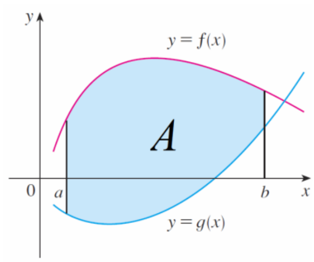


FIGURE 1
$S = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$

Chapter 6

6.1 More About Areas

Areas Between Curves

Given two curves $f(x)$ and $g(x)$ are continuous functions between $x = a$ and $x = b$ and $f(x) \geq g(x)$ for all x on $[a, b]$

$$\boxed{1}\ A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

$\boxed{2}$ So the area A of of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are continous and $f(x) \geq g(x)$ for all x in $[a, b]$

$$A = \int_a^b [f(x) - g(x)] dx$$

a and *b* could be where the points intersect or it could be the endpoint limits of integration.

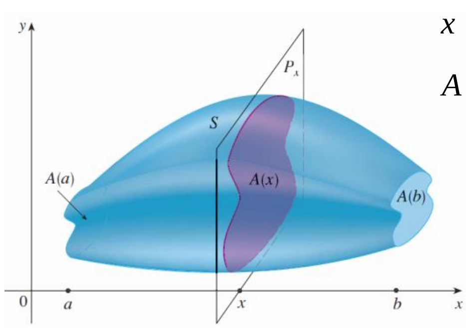
Some regions are more easily measured by treating x as a function of y.

Areas Between Parametric Curves

For a parametric function defined by the equations

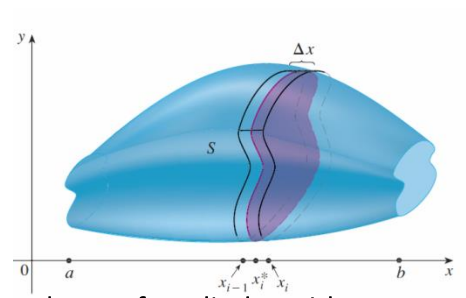
$x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$$



6.2 Volumes

Areas Between Curves



We now consider the volume of solids shapes. Recall that the volume of a cylinder with a circular base of radius r and height h is $V = Ah = \pi r^2 h$. For a solid shape *S*, we slice the shape into slabs and sum the volume of each slab, which is equal to its area multiplied by its height.

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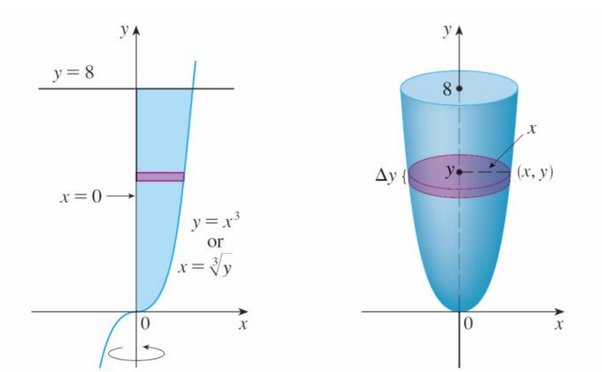
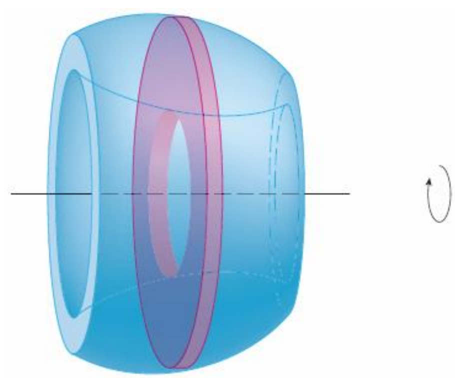
Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$.
If the cross-sectional area of S is in the plane P_x, through x and perpendicular to the x-axis, is given by $A(x)$, a continous function, then the volume of S is $V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$.
We can use discs, triangles, rectangles and more to approximate filled volumes. We utilize washers to estimate hollow shapes, called **solids of revolution**. These shapes are made by revolving a region about a line. In order to define the area of the washer, we subtract the inner area of the smaller concentric cirlce from the outer circle's total area. The formula we take for the area of the slice will depend on the shape of the cross-sectional area.





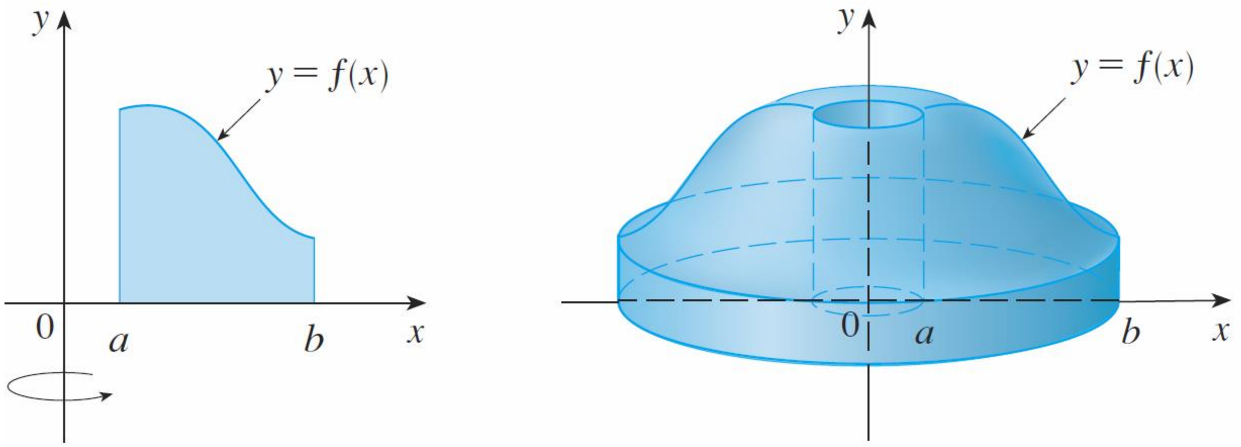
6.3 Volumes by Cylindrical Shells

Some volumes are too difficult to solve with the Washers and Discs. We turn now to the **method of cylindrical shells**. By taking to volume of concentric cylinders and subtracting the inner cylinder volume, we arrive at the definition of volume

$$
\begin{aligned}
V &= V_2 - V_1 \\
&= \pi\left(r_2^2 - r_1^2\right)h \\
&= \pi\left(r_2 - r_1\right)\left(r_2 + r_1\right)h \\
&= 2\pi \frac{\left(r_2 + r_1\right)}{2} h\left(r_2 - r_1\right)
\end{aligned}
$$

If we let $\Delta r = r_2 - r_1$ (The thickness of the shell) and $r = \frac{1}{2}\left(r_2 + r_1\right)$, then this formula for the volume of cylindrical shell becomes
$V = 2\pi r h \Delta r$ Or $V =$ [circumference][height][thickness]

Now let S be the solid obtained by rotating about the y-axis the region bounded by
$y = f(x), x = a,\ x = b,$ and the x-axis, where $b > a \geq 0$



We divide the interval $[a,b]$ into n subintervals of equal width Δx and let $\overline{x}_i$ be the midpoint of the ith subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\overline{x}_i)$ is rotated about the y-axis, then we obtain a cylindrical shell with average radius $\overline{x}_i$, height $f(\overline{x}_i)$, and thickness Δx. Its volume is

$$V_i = (2\pi\overline{x}_i)[f(\overline{x}_i)]\Delta x,$$

so an approximation of to the volume V of S is given by the sum of the volumes of these shells:

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\overline{x}_i\ f(\overline{x}_i)\,\Delta x = \int_a^b 2\pi x\ f(x)\,dx\ .$$

We conclude that the volume of a solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x\ f(x)\,dx, \text{ where } 0 \leq a < b$$

6.4 Arc Length

When we think about the length of a curve, calculus becomes useful in describing lengths of smooth curves. With polygons and simple straight lines, we can simply measure the line or use the distance formula. However, for a smooth curve, we'll need a new approach. We fit n many line segments of equal length to our smooth curve and let n go to infinity.

Arc Length Formula

If a smooth curve with parametric equations $x = f(t), y = g(t), a \le t \le b$ is traversed exactly once as t increases from a to b, then its length is given by $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Given a function $y = f(x), a \le x \le b$, we can regard x as a parameter. Then the parametric equations are $x = x, y = f(x)$, so $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

Given a function $x = f(y), a \le y \le b,$ we regard y as the parameter and the length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dx$$

6.5 Average Value of a Function

When calculating the average value of a changing function *f,* the integral is again useful.

For a discrete set of numbers with n many values in its domain, we sum the values and divide by n:

$$y_{average} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

For the average value of a curve with infinitely many values on its domain $[a,b]$, we divide the interval into n many subintervals of equal length $\Delta x = (b-a)/n$. $\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$ We can rewrite $n = (b-a)/\Delta x$, and the average value is given by $\frac{f(x_1^*) + \cdots + f(x_n^*)}{(b-a)/\Delta x}$

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{(b-a)/\Delta x} = \frac{1}{b-a}[f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

The precision of this measurement increases with n, the number of samples taken.

We let n approach infinity to write in the form of the definite integral with endpoints a and b.

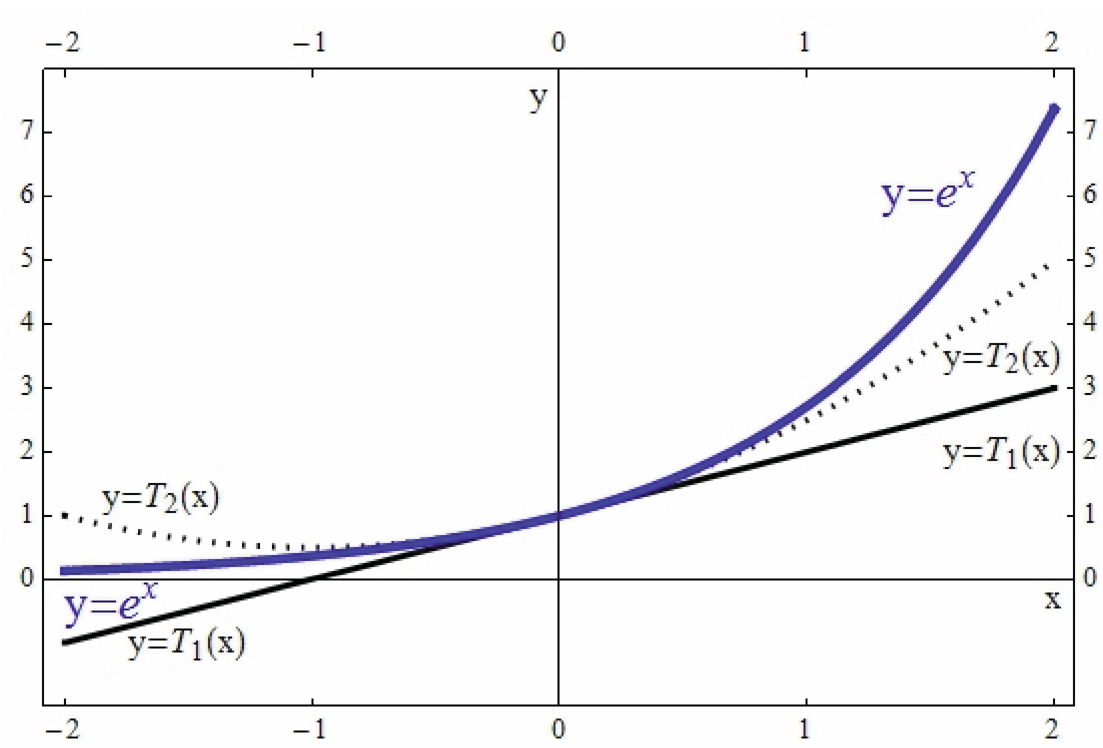
$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx = f_{average}.$$

Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{average} = \frac{1}{b-a}\int_a^b f(x)dx = f(c)(b-a)$$

The Taylor Digression



Taylor Polynomials

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$$T_n(x) = f(a) + f'(a)(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Where $n!$ is the product of the first n positive integers and 0! = 1.

We write this in summation notation as $\mathrm{T}_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

Taylor polynomial use sequential terms of the form $\frac{f^{(k)}(a)}{k!}(x-a)^k$ to approximate the true function, where the precision of the approximation increases with k, the number of terms.

Taylor's Inequality

The difference between the value of the function $f(x)$ and its Taylor $T_n(x)$ (centered at $x = a$) is called the remainder term. We define it thus: $R_n(x) = f(x) - T_n(x)$. The size of the remainder term $R_n(x)$ is an indicator of how precise an approximation of $f(x)$ the Taylor polynomial $T_n(x)$ truly is: The smaller $R_n(x)$, the closer the approximation.

The upper bounds of this error are given by $|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}$, and you choose *M.*

Infinite Series

We begin by considering the humble rational expression $\frac{1}{3}$,

whose equivalent decimal representation is the infinitely repeating $.\overline{3}$

This can be expressed as the sum of fractions $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$, if $|r| < 1$. If $|r| > 1$, $\sum_{k=0}^{\infty} ar^k$ diverges.

This never-ending addition of fractions is an example of an infinite series.

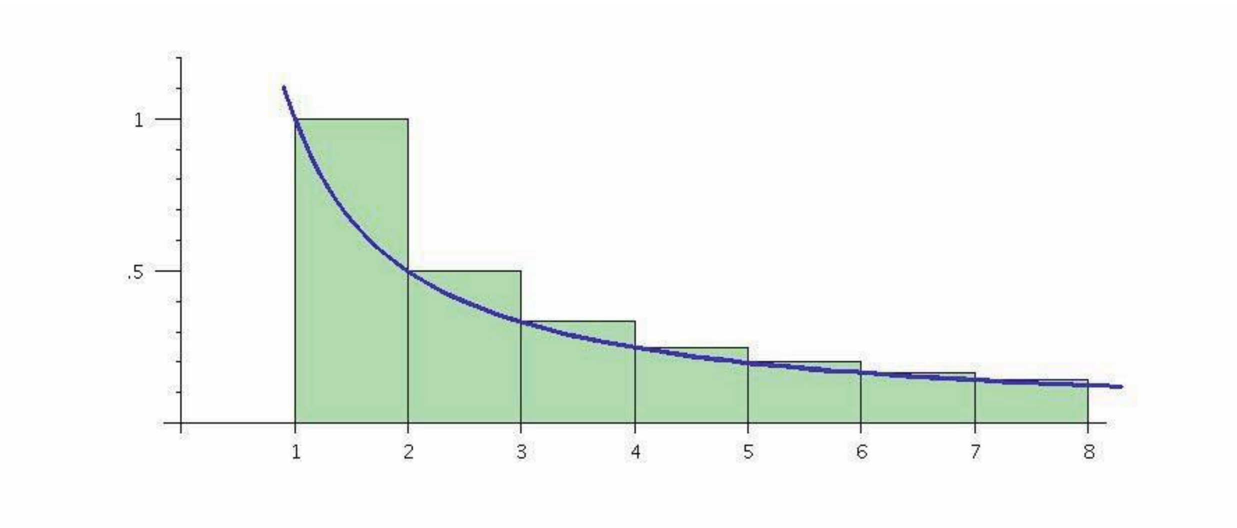
An **infinite series** is any expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$

Where $a_1, a_2, a_3, \cdots$ are $\mathbb{R}$.

Some series may converge upon a finite value**.** They are called **convergent**. $\sum_{k=1}^{\infty} k$

Others increase or decrease without bound. If a series has no finite sum we say it is **divergent**. $\sum_{k=1}^{\infty} k$

$\sum_{k=1}^{\infty} \frac{1}{k}$ is the **harmonic series**. It has no finite sum. That is, $\sum_{k=1}^{\infty} \frac{1}{k}$ is **divergent**.



Partial Sums

For any series $\sum_{k=1}^{\infty} a_k$, we define the partial sums as:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

So in general the n^{th} partial sum is $s_n = \sum_{k=1}^{\infty} a_k$

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Does this series have a sum?

If the partial sums for a series converge to a finite number, which we will denote by S, we say that the series is **convergent** and write $\sum_{k=1}^{\infty} a_k = S$. If the partial sums do not converge to any specific number, we say that the infinite series is divergent and therefore does not have a sum. Summarizing,

$$\text{If } \lim_{n\to\infty} s_n = S, \text{ then } \sum_{k=1}^{\infty} a_k = S$$

Geometric Series

A geometric series is an infinite series that has the following special form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots (a \neq 0)$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \text{if } |r| < 1. \qquad \text{If } |r| > 1, \sum_{k=0}^{\infty} ar^k \text{diverges.}$$

Taylor Series

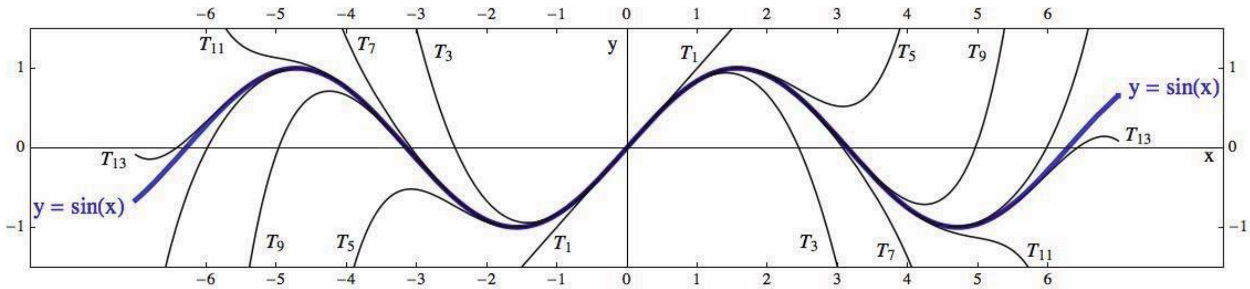
The limit of Taylor polynomials $T_n(x)$ for a function

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots, \text{ centered at } a = 0,$$

As the degree n goes to infinity.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylors approximate with higher accuracy the function as more terms are added, so taking the limit of terms to infinity we get an infinitely accurate approximation: the function itself!



Radius of Convergence

For the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for a function f(x), centered at x = a, there are only three Possibilities:

(i) The series converges only at x = a. (In this case, we say R = 0.)

(ii) The series converges absolutely for all x. (In this case, we say $R = \infty$.)

(iii) There is a positive number R for which the series converges absolutely whenever $|x-a| < R$ and diverges whenever $|x-a| > R$.

The number R (which may be 0 or infinity) is called the **radius of convergence** for the Taylor Series for $f(x)$, centered at $x = a$.

Operations with Taylor Series

We can use simple substitution with Taylor Series.

Find the Taylor series for $f(x) = \frac{1}{1+4x^2}$, find its radius of convergence,

Then show that it converges to $f(x) = \frac{1}{1+4x^2}$ whenever it converges.

We know that the Taylor series for $g(u) = \frac{1}{1-u}$ is $1 + u + u^2 + u^3 + \cdots$

And what is $f(x)$ but the composition $g(x)$ and $h(x) = -4x^2 = g(h(x)) = g(-4x^2)$? Simply substitute the composition into the series to obtain the Taylor for $f(x)$:

$$1 + \left(-4x^2\right) + \left(-4x^2\right)^2 + \left(-4x^2\right)^3 + \cdots = \sum_{j=0}^{\infty}(-4x^2)^j = \sum_{j=0}^{\infty}(-4)^j x^{2j}$$

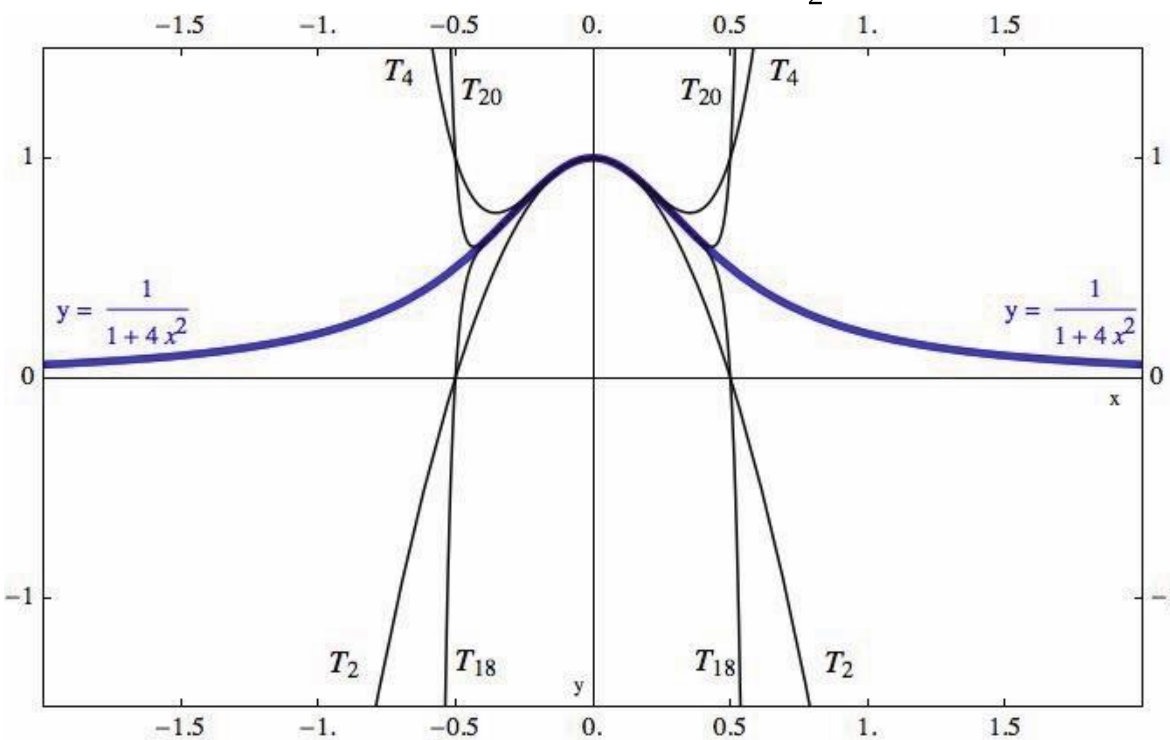
This series is geometric with $a = 1$ and $r = -4x^2$, so we find the function is convergent when

$$\frac{1}{1-r} = \frac{1}{1-(-4x^2)} \text{ if } |r| = |(-4x^2)| < 1$$

$$|-4x^2| < 1 \Leftrightarrow |x^2| < \frac{1}{4} \Leftrightarrow |x| < \frac{1}{2}$$

$$\therefore \sum_{j=0}^{\infty} (-4)^j x^{2j} = \frac{1}{1-(-4x^2)} = f(x) \text{ for } |x| < \frac{1}{2} \text{ and diverges for } |x| > \frac{1}{2}$$

We say the radius of convergence is R=$\frac{1}{2}$.



Term by Term Differentiation and Integration

We can use Taylor series to take derivatives and integrals of otherwise beastly functions.

Example: Use Taylor series to compute $\frac{d}{dx}(\sin x)$.

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + ...$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$
$$= \cos x$$

Example: find the Taylor Series for $\ln(1+x)$. We know that $\ln(1+x) = \int \frac{1}{1+x} dx, x > -1$.

We know the series expansion for $\frac{1}{1+x}$ is simply $1 - u + u^2 - u^3 + \cdots$

Substituting the series into the integral, we have

$$\int \frac{1}{1+x} dx = \int (1 - u + u^2 - u^3 + \cdots) dx \text{ for } -1 < x < 1$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ... \right) + C = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

If the center value is the same, the following share the same radius of convergence:

- The Taylor series for a function $f(x)$
- The Taylor series for its derivative $f'(x)$
- The Taylor series for its indefinite integral $\int f(x)\,dx$

Chapter 7

7.1 Modeling with Differential Equations

Models of Population Growth

We first examine population growth. We let t = time (the independent variable)

And we let P = the number of people (the dependent variable)

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The rate of growth of the population is the derivative $\frac{dP}{dt}$.

The assumption that population grows at a rate proportional to the size of itself is written as

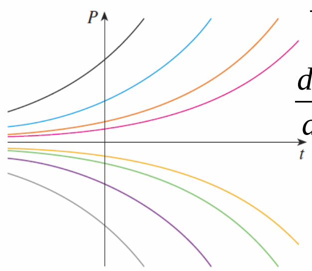


FIGURE 1
The family of solutions of $dP/dt = kP$

$\frac{dP}{dt} = kP$, where k,the proportionality constant, is held positive.

$\frac{dP}{dt} \approx kP$ if P is small (Intially, the growth rate is proportional to P.)

$\frac{dP}{dt}$<0 if $P > M$ (P decreases if it ever exceeds M.)

which we can write in one expressions as $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

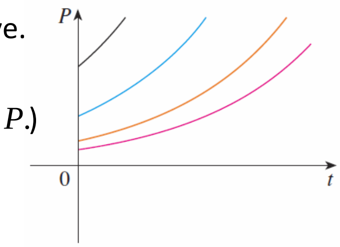


FIGURE 2
The family of solutions $P(t) = Ce^{kt}$ with $C > 0$ and $t \geqslant 0$

If we rule out negative and zero populations, our population is always increasing.

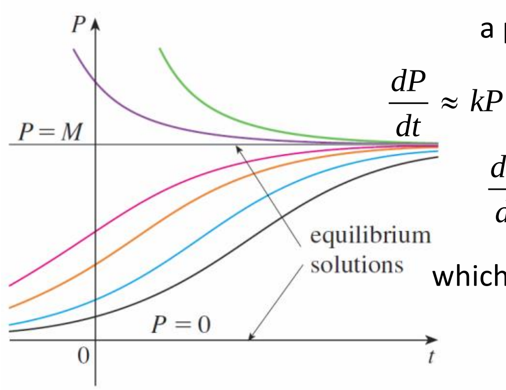
And our given derivative can be stated as $\frac{dP}{dt} = P'(t) = kP(t)$.

This states that the derivative of the function is itself multiplied by a constant k.

We know that $\frac{d}{dt}e^{kt} = k * e^{kt}$. Multiplying the expression by a coefficient C allows the expression to represent a number of solutions, called a family of solutions.

In this case, the value C represents the initial value of the population, $P(0)$, where the exponential function representing it's growth over time crosses the y-axis.

We can improve our model by accounting for realistic conditions such as limited resources. We can deduce that a population will begin growing exponentially but eventually level off approaching its *carrying capacity* M, decreasing towards M if it ever exceeds M. We now write a piecewise model:



$\frac{dP}{dt} \approx kP$ if P is small (Intially, the growth rate is proportional to P.)

$\frac{dP}{dt}$<0 if $P > M$ (P decreases if it ever exceeds M.)

which we can write in one expressions as $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

A Model for the Motion of a Spring

Next we examine Hooke's Law $F = -kx$. Newton's Second Law tells us $F = ma$ and $a = \frac{d^2x}{dt^2}$.

So we can write Hooke's Law as $m\left(\frac{d^2x}{dt^2}\right) = -kx \to \frac{d^2x}{dt^2} = \frac{-kx}{m}$.

As acceleration is the second derivative of position, this is called a *second-order differential equation.* This equations tells us the second derivative of x is proportional to x but opposite in sign. Sine and cosine have this property.

General Differential Equations

A **differential equation** is one which contains an unknown function and one or more of its derivatives. The **order** of differential equations is the order of the highest derivative.

Consider $y' = xy$ where y is an unknown function of x. A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equations. For our example, f is a solution $\Leftrightarrow f'(x) = xf(x)$.

Often we need to satisfy an additional condition, such as the **initial condition.**

Stipulations of the form $y(t_0) = y_0$ are called **initial-value problems.**

Example

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

Substituting the values $t = 0$ and $y = 2$ into the formula $y = \frac{1+ce^t}{1-ce^t}$, we have

$$2 = \frac{1+ce^0}{1-ce^0} = \frac{1+c}{1-c} \to 2 - 2c = 1 \to c = \tfrac{1}{3}$$

So we have $y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t} = \frac{3+e^t}{3-e^t}$ □

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7.2 Direction Fields and Euler's Method

It is impossible to solve most differential equations exactly, but we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method)

Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

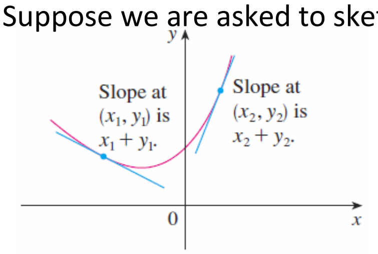


FIGURE 1
A solution of $y' = x + y$

$$y' = x + y \qquad y(0) = 1$$

Although we have no equation for the solution, we have been given a recipe for the slope at all any point (x, y). In particular, we glean that the slope at $(0,1)$ is equal to $0 + 1 = 1$

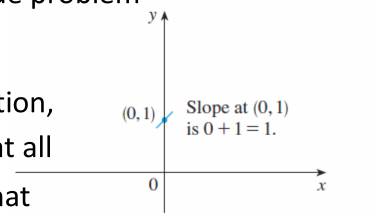


FIGURE 2
Beginning of the solution curve through (0, 1)

By sketching short line segments at a number or points with slope ($x + y$), we obtain a direction fields, which is helpful in interpolating what the solution graph should look like.

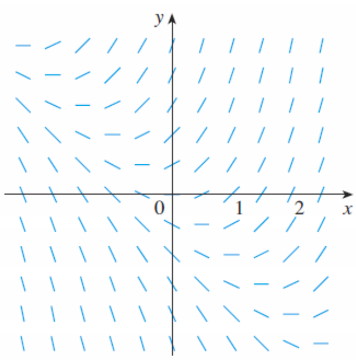


FIGURE 3
Direction field for $y' = x + y$

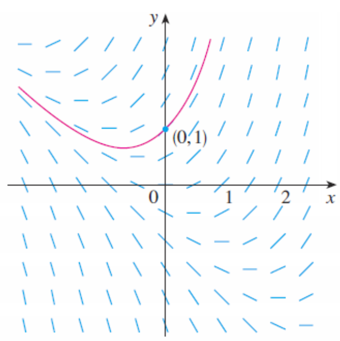
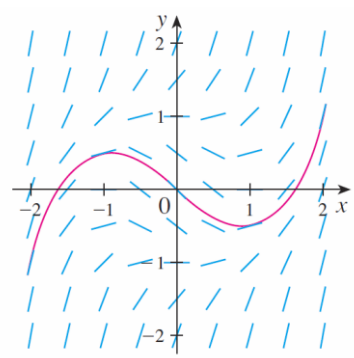


FIGURE 4
The solution curve through (0, 1)



Euler's Method

The idea here is to start at the point given by the initial value and proceed along the direction indicated by the direction field. After a short distance, look at the slope at the new location, and continue along that direction. Each stop is re-evaluation of what the slope should be based on our differential. By stopping more frequently (decreasing step size), this method yields successively more precise approximations.

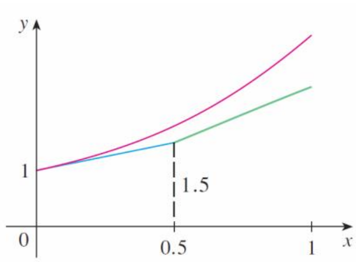


FIGURE 13
Euler approximation with step size 0.5

Euler's Method states that approximate values for the solution of the initial-value problem

$$y = F(x, y),\ y(x_0) = y_0,$$

with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad n = 1, 2, 3, \cdots$$

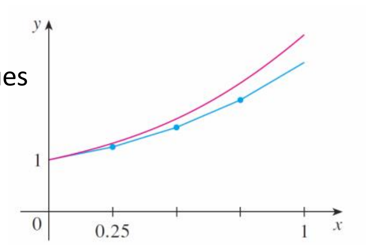


FIGURE 14
Euler approximation with step size 0.25

7.3 Separable Equations

Some differential equations can be solved explicitly. A **separable equation** is a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x multiplied by a function of y.

$$\frac{dy}{dx} = g(x)f(y)$$

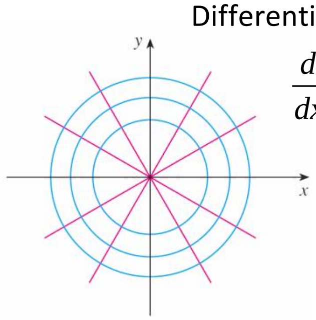
If $f(y) \neq 0$, we can write $\frac{dy}{dx} = \frac{g(x)}{h(y)}, h(y) = \frac{1}{f(y)}$.

Now we write it with x one one side and y on the other

$h(y)dy = g(x)dx$...so we can integrate both sides! $\int h(y)dy = \int g(x)dx$

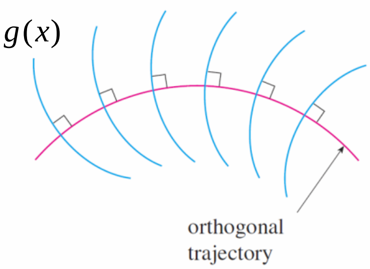
Sometime we can even solve for y in terms of x:

Differentiating implicitly on the left hand side and explicitly on the right,



$$\frac{d}{dx}\left(\int h(y)dy\right) = \frac{d}{dx}\left(\int g(x)dx\right) \therefore \frac{d}{dy}\left(\int h(y)dy\right)\frac{dy}{dx} = g(x)$$

$$\therefore h(y)\frac{dy}{dx} = g(x)$$

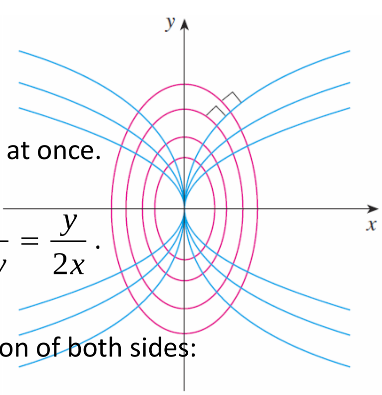


Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve at a right angle.

Find the orthogonal trajectories of the family of curves $x = ky^2$ where k is an arbitrary constant.

The curves $x = ky^2$ form a family of parabolas whose axis of symmetry is the x-axis. First, we define a differential equation that satisfies all members of the family.



Differentiating $x = ky^2$, we have $1 = 2ky\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{1}{2ky}$.

Next, we must eliminate k so that the equation is valid for all values of k at once.

We solve $k = \frac{x}{y^2}$ and write the differential equations as $\frac{dy}{dx} = \frac{1}{2\frac{x}{y^2}y} = \frac{y}{2x}$.

Having written it thus, it's a separable equation, and we can solve it with integration of both sides:

$$\int y\,dy = -\int 2x\,dx \Rightarrow \frac{y^2}{2} = -x^2 + C \Rightarrow x^2 + \frac{y^2}{2} = C$$

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Mixing Problems

Mathematical descriptions of mixing situations often lead to first-order separable differential equations.

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

With rate in = $\left(0.03\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = 0.75\frac{\text{kg}}{\text{min}}$

And rate out = $\left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$

$$\therefore \frac{dy}{dt} = .75 - \frac{y(t)}{200}\frac{\text{kg}}{\text{min}} = \frac{150 - y(t)}{200}\frac{\text{kg}}{\text{min}}$$

Again, we solve by integrating both sides

$$\int\frac{dy}{150 - y} - \int\frac{dt}{200} \Rightarrow -\ln|150 - y| = \frac{t}{200} + C$$

and $y(0) = 20$, so $-\ln 130 = C \Rightarrow \ln|150 - y| = \frac{t}{200} - \ln 130$

$|150 - y| = 130e^{-t/200} \Rightarrow$ $y(t)$ is always positive $\Rightarrow$ $y(t) = 150 - 130e^{-t/200}$

The amount of salt after 30 minutes is $y(30) = 150 - 130e^{-30/200} \approx 38.1\text{kg}$

7.4 Exponential Growth and Decay

Let $y(t)$ = the quantity y at a time t and

Let the rate of change of y with respect to t be proportional to its size $y(t)$ at any time t.

Law of natural growth: $\frac{dy}{dt} = ky, (k > 0)$ Law of natural decay: $\frac{dy}{dt} = ky, (k < 0)$

These equations are separable, so using the tools of 7.3

$$\int \frac{dy}{y} = \int k\, dt \to \ln|y| = kt + C \to |y| = e^{kt+C} = e^C e^{kt}$$

$$y = Ae^{kt} \qquad (A = \pm e^C \text{or } 0) \to y(0) = Ae^{k(0)} = A$$

$\therefore$ For the intital-value problem $\frac{dy}{dt} = ky \qquad y(0) = y_0$, the solution is $y(t) = y_0 e^{kt}$

Population Growth

$$\frac{dP}{dt} = kP \text{ or } \frac{1}{P}\frac{dP}{dt} = k$$

The quantity $\frac{1}{P}\frac{dP}{dt}$ is called the **relative growth rate**. Instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation with relative decay rate $-\frac{1}{m}\frac{dm}{dt}$.

The rate of decay is expressed in terms of **half-life**, the time required for half of the given quantity to decay.

$$\frac{dm}{dt} = km \qquad m(t) = m_0 e^{kt}$$

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings provided the difference is not too large.

$\frac{dT}{dt} = k(T = T_s)$ Change of variable $\frac{dy}{dt} = ky$

Continuously Compounded Interest

If an amount A_0 is invested at an interest rate r, then after t years it's worth $A_0(1+r)^t$.

With interest that compound more or less frequently, n times per year, the value is $A_0\left(1+\frac{r}{n}\right)^{nt}$.

For continuously compounding interest,

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = \lim_{n\to\infty} A_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = A_0\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \text{(where } m = n/r) = A_0e^{rt}$$

7.5 The Logistic Equation

The Logistic Model

Building on our model of growth, we now incorporate a carrying capacity M and call it

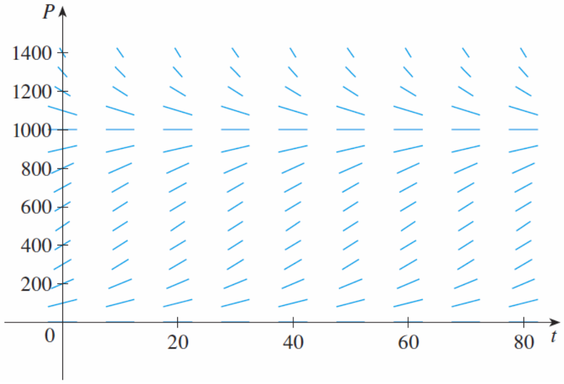
the **logistic differential equation**:

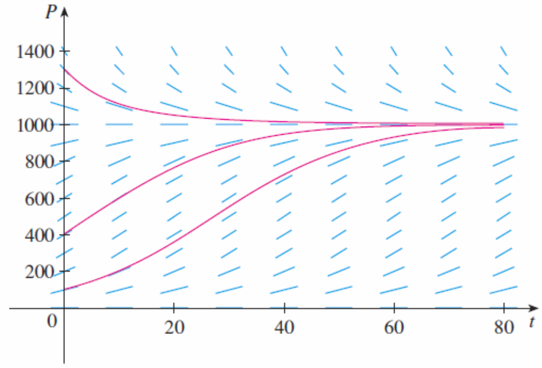
$$\frac{1}{P}\frac{dP}{dt} = k\left(1-\frac{P}{M}\right) \to \frac{dP}{dt} = kP\left(1-\frac{P}{M}\right)$$

This behavior implies that P will approach M from above or below when non-zero.

Direction Fields

Direction fields like this one demonstrate the idea of carry capacity





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Euler's Method

Example: Use Euler's method with step sizes 20, 10, 5, 1, and .1 to estimate the population sizes $P(40)$ and $P(80)$, where P is the solution of the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

SOLUTION With step size $h = 20$, $t_0 = 0$, $P_0 = 100$, and

$$F(t, P) = 0.08P\left(1 - \frac{P}{1000}\right)$$

we get, using the notation of Section 7.2,

$$\begin{aligned}
t = 20{:}\quad & P_1 = 100 + 20F(0, 100) = 244 \\
t = 40{:}\quad & P_2 = 244 + 20F(20, 244) \approx 539.14 \\
t = 60{:}\quad & P_3 = 539.14 + 20F(40, 539.14) \approx 936.69 \\
t = 80{:}\quad & P_4 = 936.69 + 20F(60, 936.69) \approx 1031.57
\end{aligned}$$

Thus our estimates for the population sizes at times $t = 40$ and $t = 80$ are

$$P(40) \approx 539 \qquad P(80) \approx 1032$$

The Analytic Solution

The logistic equation is separable and thus explicitly solvable.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \rightarrow \int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the LHS, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$ and use partial fractions to get $$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

Thus

$$\int\left(\frac{1}{P} + \frac{1}{M - P}\right)dP = \int k\,dt \qquad \rightarrow \ln|P| - \ln|M - P| = kt + C \qquad \rightarrow \ln\left|\frac{M - P}{P}\right| = -kt - C$$

$$\rightarrow \left|\frac{M - P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt} \qquad \rightarrow \frac{M - P}{P} = Ae^{-kt}$$

The solution to the logistic equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$ where $A = \dfrac{M - P_0}{P_0}$